

# SECOND ORDER NORMALIZATION IN THE GENERALIZED PHOTOGRAVITATIONAL RESTRICTED THREE BODY PROBLEM WITH POYNTING-ROBERTSON DRAG

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**ABSTRACT.** In this paper we have performed second order normalization in the generalized photogravitational restricted three body problem with Poynting-Robertson drag. We have performed Birkhoff's normalization of the Hamiltonian. For this we have utilised Henrard's method and expanded the coordinates of the third body in Double d'Alembert series. We have found the values of first and second order components. The second order components are obtained as solutions of the two partial differential equations. We have employed the first condition of KAM theorem in solving these equations. The first and second order components are affected by radiation pressure, oblateness and P-R drag. Finally we obtained the third order part  $H_3$  of the Hamiltonian in  $I_1^{1/2} I_2^{1/2}$  zero.

**AMS Classification:** 70F15

**Keywords:** Second Order Normalization/Generalized Photogravitational/RTBP/P-R drag.

## 1. INTRODUCTION

The restricted three body problem describes the motion of an infinitesimal mass moving under the gravitational effect of the two finite masses, called primaries, which move in circular orbits around their centre of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The classical restricted three body problem is generalized to include the force of radiation pressure, the Poynting-Robertson effect and oblateness effect.

J. H. Poynting (1903) considered the effect of the absorption and subsequent re-emission of sunlight by small isolated particles in the solar system. His work was later modified by H. P. Robertson (1937) who used a precise relativistic treatments of the first order in the ratio of the velocity of the particle to that of light.

The effect of radiation pressure and P-R. drag in the restricted three body problem has been studied by Colombo *et al.* (1966), Chernikov Yu. A. (1970) and Schuerman (1980) who discussed the position as well as the stability of the Lagrangian equilibrium points when radiation pressure, P-R drag force are included. Murray C. D. (1994) systematically discussed the dynamical effect of general drag in the planar circular restricted three body problem, Liou J. C. *et al.* (1995) examined the effect of radiation pressure, P-R drag and solar wind drag in the restricted three body problem.

Moser's conditions (1962), Arnold's theorem (1961) and Liapunov's theorem (1956) played a significant role in deciding the nonlinear stability of an equilibrium point. Applying Arnold's theorem (1961), Leontovic (1962) examined the nonlinear stability of triangular points. Moser gave some modifications in Arnold's theorem. Then Deprit and Deprit (1967) investigated the nonlinear stability of triangular points by applying Moser's modified version of Arnold's theorem (1961).

Bhatnagar and Hallan (1983) studied the effect of perturbations on the nonlinear stability of triangular points. Maciejewski and Gozdziowski (1991) described the normalization algorithms of Hamiltonian near an equilibrium point. Niedzielska (1994) investigated the nonlinear stability of the libration points in the photogravitational restricted three body problem. Mishra P. and Ishwar B. (1995) studied second order normalization in the generalized restricted problem of three bodies, smaller primary being an oblate spheroid. Ishwar B. (1997) studied nonlinear stability in the generalized restricted three body problem.

In this paper we have performed Birkhoff's normalization of the Hamiltonian. For this we have utilised Henrard's method and expanded the coordinates of the third body in Double d'Alembert series. We have found the values of first and second order components. The second order components are obtained as solutions of the two partial differential equations. We have employed the first condition of KAM theorem in solving these equations. The first and second order components are affected by radiation pressure, oblateness and P-R drag. Finally we obtained the third order part  $H_3$  of the Hamiltonian in  $I_1^{1/2}I_2^{1/2}$  zero.

## 2. LOCATION OF TRIANGULAR EQUILIBRIUM POINTS

Equations of motions are

$$\begin{aligned} (1) \quad & \ddot{x} - 2n\dot{y} = U_x, \quad \text{where,} \quad U_x = \frac{\partial U_1}{\partial x} - \frac{W_1 N_1}{r_1^2} \\ (2) \quad & \ddot{y} + 2n\dot{x} = U_y, \quad U_y = \frac{\partial U_1}{\partial y} - \frac{W_1 N_2}{r_1^2} \\ (3) \quad & U_1 = \frac{n^2(x^2 + y^2)}{2} + \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} \end{aligned}$$

$$\begin{aligned} r_1^2 &= (x + \mu)^2 + y^2, \quad r_2^2 = (x + \mu - 1)^2 + y^2, \quad n^2 = 1 + \frac{3}{2}A_2, \\ N_1 &= \frac{(x + \mu)[(x + \mu)\dot{x} + y\dot{y}]}{r_1^2} + \dot{x} - ny, \quad N_2 = \frac{y[(x + \mu)\dot{x} + y\dot{y}]}{r_1^2} + \dot{y} + n(x + \mu) \end{aligned}$$

$W_1 = \frac{(1-\mu)(1-q_1)}{c_d}$ ,  $\mu = \frac{m_2}{m_1+m_2} \leq \frac{1}{2}$ ,  $m_1, m_2$  be the masses of the primaries,  $A_2 = \frac{r_e^2 - r_p^2}{5r^2}$  be the oblateness coefficient,  $r_e$  and  $r_p$  be the equatorial and polar radii respectively  $r$  be the distance between primaries,  $q = (1 - \frac{F_p}{F_g})$  be the mass reduction factor expressed in terms of the particle's radius  $a$ , density  $\rho$  and radiation pressure efficiency factor  $\chi$  (in the C.G.S.system) i.e.,  $q = 1 - \frac{5.6 \times 10^{-5} \chi}{a\rho}$ . Assumption  $q = \text{constant}$  is equivalent to neglecting fluctuation in the beam of solar radiation and the effect of solar radiation, the effect of the planet's shadow, obviously  $q \leq 1$ . Triangular equilibrium points are given by  $U_x = 0, U_y = 0, z = 0, y \neq 0$ , then we have

$$\begin{aligned} (4) \quad & x_* = x_0 \left\{ 1 - \frac{nW_1 \left[ (1-\mu) \left( 1 + \frac{5}{2}A_2 \right) + \mu \left( 1 - \frac{A_2}{2} \right) \frac{\delta^2}{2} \right]}{3\mu(1-\mu)y_0x_0} - \frac{\delta^2}{2} \frac{A_2}{x_0} \right\} \\ (5) \quad & y_* = y_0 \left\{ 1 - \frac{nW_1 \delta^2 \left[ 2\mu - 1 - \mu \left( 1 - \frac{3A_2}{2} \right) \frac{\delta^2}{2} + 7(1-\mu) \frac{A_2}{2} \right]}{3\mu(1-\mu)y_0^3} - \frac{\delta^2 \left( 1 - \frac{\delta^2}{2} \right) A_2}{y_0^2} \right\}^{1/2} \end{aligned}$$

where  $x_0 = \frac{\delta^2}{2} - \mu$ ,  $y_0 = \pm \delta \left( 1 - \frac{\delta^2}{4} \right)^{1/2}$  and  $\delta = q_1^{1/3}$ , as in preprint Kushvah & Ishwar (2006)

## 3. SECOND ORDER NORMALIZATION

The Lagrangian function of the problem can be written as

$$\begin{aligned} (6) \quad & L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2 + y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} \\ & + W_1 \left\{ \frac{(x + \mu)\dot{x} + y\dot{y}}{2r_1^2} - n \arctan \frac{y}{(x + \mu)} \right\} \end{aligned}$$

and the Hamiltonian is  $H = -L + P_x \dot{x} + P_y \dot{y}$ , where  $P_x, P_y$  are the momenta coordinates given by

$$P_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - ny + \frac{W_1}{2r_1^2}(x + \mu), \quad P_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + nx + \frac{W_1}{2r_1^2}y$$

For simplicity we suppose  $q_1 = 1 - \epsilon$ , with  $|\epsilon| \ll 1$  then coordinates of triangular equilibrium points can be written in the form

$$(7) \quad x = \frac{\gamma}{2} - \frac{\epsilon}{3} - \frac{A_2}{2} + \frac{A_2\epsilon}{3} - \frac{(9+\gamma)}{6\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1$$

$$(8) \quad y = \frac{\sqrt{3}}{2} \left\{ 1 - \frac{2\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} + \frac{(1+\gamma)}{9\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1 \right\}$$

where  $\gamma = 1 - 2\mu$ . We shift the origin to  $L_4$ . For that, we change  $x \rightarrow x_* + x$  and  $y \rightarrow y_* + y$ . Let  $a = x_* + \mu, b = y_*$  so that

$$(9) \quad a = \frac{1}{2} \left\{ -\frac{2\epsilon}{3} - A_2 + \frac{2A_2\epsilon}{3} - \frac{(9+\gamma)}{3\sqrt{3}}nW_1 - \frac{8\gamma\epsilon}{27\sqrt{3}}nW_1 \right\}$$

$$(10) \quad b = \frac{\sqrt{3}}{2} \left\{ 1 - \frac{2\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} + \frac{(1+\gamma)}{9\sqrt{3}}nW_1 - \frac{4\gamma\epsilon}{27\sqrt{3}}nW_1 \right\}$$

Expanding  $L$  in power series of  $x$  and  $y$ , we get

$$(11) \quad L = L_0 + L_1 + L_2 + L_3 + \dots$$

$$(12) \quad H = H_0 + H_1 + H_2 + H_3 + \dots = -L + P_x \dot{x} + P_y \dot{y}$$

where  $L_0, L_1, L_2, L_3 \dots$  are constants, first order term, second order term, ... respectively. Third order term  $H_3$  of Hamiltonian can be written as

$$(13) \quad H_3 = -L_3 = -\frac{1}{3!} \{ x^3 T_1 + 3x^2 y T_2 + 3xy^2 T_3 + y^3 T_4 + 6T_5 \}$$

where

$$(14) \quad T_1 = \frac{3}{16} \left[ \frac{16}{3}\epsilon + 6A_2 - \frac{979}{18}A_2\epsilon + \frac{(143+9\gamma)}{6\sqrt{3}}nW_1 + \frac{(459+376\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\ \left. + \gamma \left\{ 14 + \frac{4\epsilon}{3} + 25A_2 - \frac{1507}{18}A_2\epsilon - \frac{(215+29\gamma)}{6\sqrt{3}}nW_1 - \frac{2(1174+169\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right]$$

$$(15) \quad T_2 = \frac{3\sqrt{3}}{16} \left[ 14 - \frac{16}{3}\epsilon + \frac{A_2}{3} - \frac{367}{18}A_2\epsilon + \frac{115(1+\gamma)}{18\sqrt{3}}nW_1 - \frac{(959-136\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\ \left. + \gamma \left\{ \frac{32\epsilon}{3} + 40A_2 - \frac{382}{9}A_2\epsilon + \frac{(511+53\gamma)}{6\sqrt{3}}nW_1 - \frac{(2519-24\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right]$$

$$(16) \quad T_3 = \frac{-9}{16} \left[ \frac{8}{3}\epsilon + \frac{203A_2}{6} - \frac{625}{54}A_2\epsilon - \frac{(105+15\gamma)}{18\sqrt{3}}nW_1 - \frac{(403-114\gamma)}{81\sqrt{3}}nW_1\epsilon \right. \\ \left. + \gamma \left\{ 2 - \frac{4\epsilon}{9} + \frac{55A_2}{2} - \frac{797}{54}A_2\epsilon + \frac{(197+23\gamma)}{18\sqrt{3}}nW_1 - \frac{(211-32\gamma)}{81\sqrt{3}}nW_1\epsilon \right\} \right]$$

$$(17) \quad T_4 = \frac{-9\sqrt{3}}{16} \left[ 2 - \frac{8}{3}\epsilon + \frac{23A_2}{3} - 44A_2\epsilon - \frac{(37+\gamma)}{18\sqrt{3}}nW_1 - \frac{(219+253\gamma)}{81\sqrt{3}}nW_1\epsilon \right. \\ \left. + \gamma \left\{ 4\epsilon + \frac{88}{27}A_2\epsilon + \frac{(241+45\gamma)}{18\sqrt{3}}nW_1 - \frac{(1558-126\gamma)}{81\sqrt{3}}nW_1\epsilon \right\} \right]$$

$$(18) \quad T_5 = \frac{W_1}{2(a^2+b^2)^3} \left[ (a\dot{x} + b\dot{y}) \{ 3(ax+by) - (bx-ay)^2 \} - 2(x\dot{x} + y\dot{y})(ax+by)(a^2+b^2) \right]$$

In order to perform Birkhoff's normalization, we use Henrard's method [Deprit and Deprit Brtholomé (1967)] for which the coordinates  $(x, y)$  of infinitesimal body, to be expanded in double d'Alembert series  $x = \sum_{n \geq 1} B_n^{1,0}, y = \sum_{n \geq 1} B_n^{0,1}$  where the homogeneous components  $B_n^{1,0}$  and  $B_n^{0,1}$  of degree  $n$  are of the form

$$(19) \quad \sum_{0 \leq m \leq n} I_1^{\frac{n-m}{2}} I_2^{\frac{m}{2}} \sum_{(p,q)} C_{n-m,m,p,q} \cos(p\phi_1 + q\phi_2) + S_{n-m,m,p,q} \sin(p\phi_1 + q\phi_2)$$

The condition in double summation are (i)  $p$  runs over those integers in the interval  $0 \leq p \leq n - m$  that have the same parity as  $n - m$  (ii)  $q$  runs over those integers in the interval  $-m \leq q \leq m$  that have the same parity as  $m$ . Here  $I_1, I_2$  are the action momenta coordinates which are to be taken as constants of integer,  $\phi_1, \phi_2$  are angle coordinates to be determined as linear functions of time in such a way that  $\dot{\phi}_1 = \omega_1 + \sum_{n \geq 1} f_{2n}(I_1, I_2), \dot{\phi}_2 = \omega_2 + \sum_{n \geq 1} g_{2n}(I_1, I_2)$  where  $\omega_1, \omega_2$  are the basic frequencies,  $f_{2n}$  and  $g_{2n}$  are of the form

$$(20) \quad f_{2n} = \sum_{0 \leq m \leq n} f'_{2(n-m), 2m} I_1^{n-m} I_2^m$$

$$(21) \quad g_{2n} = \sum_{0 \leq m \leq n} g'_{2(n-m), 2m} I_1^{n-m} I_2^m$$

The first order components  $B_1^{1,0}$  and  $B_1^{0,1}$  in  $I_1, I_2$  are the values of  $x$  and  $y$  given by

$$(22) \quad X = JT \quad \text{where} \quad X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix}, J = [J_{ij}]_{1 \leq i \leq j \leq 4}, T = \begin{bmatrix} Q_1 \\ Q_2 \\ p_1 \\ p_2 \end{bmatrix}$$

$$(23) \quad P_i = (2I_i \omega_i)^{1/2} \cos \phi_i, \quad Q_i = \left(\frac{2I_i}{\omega_i}\right)^{1/2} \sin \phi_i, \quad (i = 1, 2)$$

$$(24) \quad B_1^{1,0} = J_{13} \sqrt{2\omega_1 I_1} \cos \phi_1 + J_{14} \sqrt{2\omega_2 I_2} \cos \phi_2$$

$$(25) \quad B_1^{0,1} = J_{21} \sqrt{\frac{2I_1}{\omega_1}} \sin \phi_1 + J_{22} \sqrt{\frac{2I_2}{\omega_2}} \sin \phi_2 + J_{23} \sqrt{2I_1 \omega_1} \cos \phi_1 + J_{24} \sqrt{2I_2 \omega_2} \sin \phi_2$$

where

$$(26) \quad J_{13} = \frac{l_1}{2\omega_1 k_1} \left\{ 1 - \frac{1}{2l_1^2} \left[ \epsilon + \frac{45A_2}{2} - \frac{717A_2\epsilon}{36} + \frac{(67+19\gamma)}{12\sqrt{3}} nW_1 - \frac{(431-3\gamma)}{27\sqrt{3}} nW_1\epsilon \right] \right. \\ + \frac{\gamma}{2l_1^2} \left[ 3\epsilon - \frac{29A_2}{36} - \frac{(187+27\gamma)}{12\sqrt{3}} nW_1 - \frac{2(247+3\gamma)}{27\sqrt{3}} nW_1\epsilon \right] \\ - \frac{1}{2k_1^2} \left[ \frac{\epsilon}{2} - 3A_2 - \frac{73A_2\epsilon}{24} + \frac{(1-9\gamma)}{24\sqrt{3}} nW_1 + \frac{(53-39\gamma)}{54\sqrt{3}} nW_1\epsilon \right] \\ - \frac{\gamma}{4k_1^2} \left[ \epsilon - 3A_2 - \frac{299A_2\epsilon}{72} - \frac{(6-5\gamma)}{12\sqrt{3}} nW_1 - \frac{(266-93\gamma)}{54\sqrt{3}} nW_1\epsilon \right] \\ \left. + \frac{\epsilon}{4l_1^2 k_1^2} \left[ \frac{3A_2}{4} + \frac{(33+14\gamma)}{12\sqrt{3}} nW_1 \right] + \frac{\gamma\epsilon}{8l_1^2 k_1^2} \left[ \frac{347A_2}{36} - \frac{(43-8\gamma)}{4\sqrt{3}} nW_1 \right] \right\}$$

$$(27) \quad J_{14} = \frac{l_2}{2\omega_2 k_2} \left\{ 1 - \frac{1}{2l_2^2} \left[ \epsilon + \frac{45A_2}{2} - \frac{717A_2\epsilon}{36} + \frac{(67+19\gamma)}{12\sqrt{3}} nW_1 - \frac{(431-3\gamma)}{27\sqrt{3}} nW_1\epsilon \right] \right. \\ - \frac{\gamma}{2l_2^2} \left[ 3\epsilon - \frac{293A_2}{36} + \frac{(187+27\gamma)}{12\sqrt{3}} nW_1 - \frac{2(247+3\gamma)}{27\sqrt{3}} nW_1\epsilon \right] \\ - \frac{1}{2k_2^2} \left[ \frac{\epsilon}{2} - 3A_2 - \frac{73A_2\epsilon}{24} + \frac{(1-9\gamma)}{24\sqrt{3}} nW_1 + \frac{(53-39\gamma)}{54\sqrt{3}} nW_1\epsilon \right] \\ + \frac{\gamma}{2k_2^2} \left[ \epsilon - 3A_2 - \frac{299A_2\epsilon}{72} - \frac{(6-5\gamma)}{12\sqrt{3}} nW_1 - \frac{(268-9\gamma)}{54\sqrt{3}} nW_1\epsilon \right] \\ \left. - \frac{\epsilon}{4l_2^2 k_2^2} \left[ \frac{33A_2}{4} + \frac{(1643-93\gamma)}{216\sqrt{3}} nW_1 \right] + \frac{\gamma\epsilon}{4l_2^2 k_2^2} \left[ \frac{737A_2}{72} - \frac{(13+2\gamma)}{\sqrt{3}} nW_1 \right] \right\}$$

$$\begin{aligned}
(28) \quad J_{21} = & -\frac{4n\omega_1}{l_1 k_1} \left\{ 1 + \frac{1}{2l_1^2} \left[ \epsilon + \frac{45A_2}{2} - \frac{717A_2\epsilon}{36} + \frac{(67+19\gamma)}{12\sqrt{3}}nW_1 - \frac{(413-3\gamma)}{27\sqrt{3}}nW_1\epsilon \right] \right. \\
& - \frac{\gamma}{2l_1^2} \left[ 3\epsilon - \frac{293A_2}{36} + \frac{(187+27\gamma)}{12\sqrt{3}}nW_1 - \frac{2(247+3\gamma)}{27\sqrt{3}}nW_1\epsilon \right] \\
& - \frac{1}{2k_1^2} \left[ \frac{\epsilon}{2} - 3A_2 - \frac{73A_2\epsilon}{24} + \frac{(1-9\gamma)}{24\sqrt{3}}nW_1 + \frac{(53-39\gamma)}{54\sqrt{3}}nW_1\epsilon \right] \\
& - \frac{\gamma}{4k_1^2} \left[ \epsilon - 3A_2 - \frac{299A_2\epsilon}{72} - \frac{(6-5\gamma)}{12\sqrt{3}}nW_1 - \frac{(268-93\gamma)}{54\sqrt{3}}nW_1\epsilon \right] \\
& \left. + \frac{\epsilon}{8l_1^2 k_1^2} \left[ \frac{33A_2}{4} + \frac{(68-10\gamma)}{24\sqrt{3}}nW_1 \right] + \frac{\gamma\epsilon}{8l_1^2 k_1^2} \left[ \frac{242A_2}{9} + \frac{(43-8\gamma)}{4\sqrt{3}}nW_1 \right] \right\}
\end{aligned}$$

$$\begin{aligned}
(29) \quad J_{22} = & \frac{4n\omega_2}{l_2 k_2} \left\{ 1 + \frac{1}{2l_2^2} \left[ \epsilon + \frac{45A_2}{2} - \frac{717A_2\epsilon}{36} + \frac{(67+19\gamma)}{12\sqrt{3}}nW_1 - \frac{(413-3\gamma)}{27\sqrt{3}}nW_1\epsilon \right] \right. \\
& - \frac{\gamma}{2l_2^2} \left[ 3\epsilon - \frac{293A_2}{36} + \frac{(187+27\gamma)}{12\sqrt{3}}nW_1 - \frac{2(247+3\gamma)}{27\sqrt{3}}nW_1\epsilon \right] \\
& + \frac{1}{2k_2^2} \left[ \frac{\epsilon}{2} - 3A_2 - \frac{73A_2\epsilon}{24} + \frac{(1-9\gamma)}{24\sqrt{3}}nW_1 + \frac{(53-39\gamma)}{54\sqrt{3}}nW_1\epsilon \right] \\
& - \frac{\gamma}{4k_2^2} \left[ \epsilon - 3A_2 - \frac{299A_2\epsilon}{72} - \frac{(6-5\gamma)}{12\sqrt{3}}nW_1 - \frac{(268-93\gamma)}{54\sqrt{3}}nW_1\epsilon \right] \\
& \left. + \frac{\epsilon}{4l_2^2 k_2^2} \left[ \frac{33A_2}{4} + \frac{(34+5\gamma)}{12\sqrt{3}}nW_1 \right] + \frac{\gamma\epsilon}{8l_2^2 k_2^2} \left[ \frac{75A_2}{2} + \frac{(43-8\gamma)}{4\sqrt{3}}nW_1 \right] \right\}
\end{aligned}$$

$$\begin{aligned}
(30) \quad J_{23} = & \frac{\sqrt{3}}{4\omega_1 l_1 k_1} \left\{ 2\epsilon + 6A_2 + \frac{37A_2\epsilon}{2} - \frac{(13+\gamma)}{2\sqrt{3}}nW_1 + \frac{2(79-7\gamma)}{9\sqrt{3}}nW_1\epsilon \right. \\
& - \gamma \left[ 6 + \frac{2\epsilon}{3} + 13A_2 - \frac{33A_2\epsilon}{2} + \frac{(11-\gamma)}{2\sqrt{3}}nW_1 - \frac{(186-\gamma)}{9\sqrt{3}}nW_1\epsilon \right] \\
& + \frac{1}{2l_1^2} \left[ 51A_2 + \frac{(14+8\gamma)}{3\sqrt{3}}nW_1 \right] - \frac{\epsilon}{k_1^2} \left[ 3A_2 + \frac{(19+6\gamma)}{6\sqrt{3}}nW_1 \right] \\
& - \frac{\gamma}{2l_1^2} \left[ 6\epsilon + 135A_2 - \frac{808A_2\epsilon}{9} - \frac{(67+19\gamma)}{2\sqrt{3}}nW_1 - \frac{(755+19\gamma)}{9\sqrt{3}}nW_1\epsilon \right] \\
& - \frac{\gamma}{2k_1^2} \left[ 3\epsilon - 18A_2 - \frac{55A_2\epsilon}{4} - \frac{(1-9\gamma)}{4\sqrt{3}}nW_1 + \frac{(923-60\gamma)}{12\sqrt{3}}nW_1\epsilon \right] \\
& \left. + \frac{\gamma\epsilon}{8l_1^2 k_1^2} \left[ \frac{9A_2}{2} + \frac{(34-5\gamma)}{2\sqrt{3}}nW_1 \right] \right\}
\end{aligned}$$

$$\begin{aligned}
(31) \quad J_{24} = & \frac{\sqrt{3}}{4\omega_2 l_2 k_2} \left\{ 2\epsilon + 6A_2 + \frac{37A_2\epsilon}{2} - \frac{(13+\gamma)}{2\sqrt{3}}nW_1 + \frac{2(79-7\gamma)}{9\sqrt{3}}nW_1\epsilon \right. \\
& - \gamma \left[ 6 + \frac{2\epsilon}{3} + 13A_2 - \frac{33A_2\epsilon}{2} + \frac{(11-\gamma)}{2\sqrt{3}}nW_1 - \frac{(186-\gamma)}{9\sqrt{3}}nW_1\epsilon \right] \\
& - \frac{1}{2l_2^2} \left[ 51A_2 + \frac{(14+8\gamma)}{3\sqrt{3}}nW_1 \right] - \frac{\epsilon}{k_2^2} \left[ 3A_2 + \frac{(19+6\gamma)}{6\sqrt{3}}nW_1 \right] \\
& - \frac{\gamma}{2l_2^2} \left[ 6\epsilon + 135A_2 - \frac{808A_2\epsilon}{9} - \frac{(67+19\gamma)}{2\sqrt{3}}nW_1 - \frac{(755+19\gamma)}{9\sqrt{3}}nW_1\epsilon \right] \\
& - \frac{\gamma}{2k_1^2} \left[ 3\epsilon - 18A_2 - \frac{55A_2\epsilon}{4} - \frac{(1-9\gamma)}{4\sqrt{3}}nW_1 + \frac{(923-60\gamma)}{12\sqrt{3}}nW_1\epsilon \right] \\
& \left. - \frac{\gamma\epsilon}{4l_1^2 k_1^2} \left[ \frac{99A_2}{2} + \frac{(34-5\gamma)}{2\sqrt{3}}nW_1 \right] \right\}
\end{aligned}$$

with  $l_j^2 = 4\omega_j^2 + 9$ , ( $j = 1, 2$ ) and  $k_1^2 = 2\omega_1^2 - 1$ ,  $k_2^2 = -2\omega_2^2 + 1$ . In order to findout the second order components  $B_2^{1,0}, B_2^{0,1}$  we consider Lagrange's equations of motion

$$(32) \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0$$

$$(33) \quad \text{i.e.} \quad \left. \begin{aligned} \ddot{x} - 2n\dot{y} + (2E - n^2)x + Gy &= \frac{\partial L_3}{\partial x} + \frac{\partial L_4}{\partial x} \\ \ddot{y} + 2n\dot{x} + (2F - n^2)y + Gx &= \frac{\partial L_3}{\partial y} + \frac{\partial L_4}{\partial y} \end{aligned} \right\}$$

Since  $x$  and  $y$  are double d'Alembert series,  $x^j x^k$  ( $j \geq 0, k \geq 0, j + k \geq 0$ ) is also a double d'Alembert series, the time derivatives  $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}$  are also double d'Alembert series. We can write

$$\dot{x} = \sum_{n \geq 1} \dot{x}_n, \quad \dot{y} = \sum_{n \geq 1} \dot{y}_n, \quad \ddot{x} = \sum_{n \geq 1} \ddot{x}_n, \quad \ddot{y} = \sum_{n \geq 1} \ddot{y}_n$$

where  $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}$  are homogeneous components of degree  $n$  in  $I_1^{1/2}, I_2^{1/2}$  i.e.

$$(34) \quad \dot{x} = \frac{d}{dt} \sum_{n \geq 1} B_n^{1,0} = \sum_{n \geq 1} \left[ \frac{\partial B_n^{1,0}}{\partial \phi_1} (\omega_1 + f_2 + f_4 + \dots) + \frac{\partial B_n^{1,0}}{\partial \phi_2} (-\omega_2 + g_2 + g_4 + \dots) \right]$$

We write three components  $\dot{x}_1, \dot{x}_2, \dot{x}_3$  of  $\dot{x}$

$$(35) \quad \dot{x}_1 = \omega_1 \frac{\partial B_1^{1,0}}{\partial \phi_1} - \omega_2 \frac{\partial B_1^{1,0}}{\partial \phi_2} = D B_1^{1,0}$$

$$(36) \quad \dot{x}_2 = \omega_1 \frac{\partial B_2^{1,0}}{\partial \phi_1} - \omega_2 \frac{\partial B_2^{1,0}}{\partial \phi_2} = D B_2^{1,0}$$

$$(37) \quad \begin{aligned} \dot{x}_3 &= \omega_1 \frac{\partial B_3^{1,0}}{\partial \phi_1} - \omega_2 \frac{\partial B_3^{1,0}}{\partial \phi_2} + f_2 \frac{\partial B_1^{1,0}}{\partial \phi_1} - g_2 \frac{\partial B_1^{1,0}}{\partial \phi_2} \\ &= D B_2^{1,0} + f_2 \frac{\partial B_1^{1,0}}{\partial \phi_1} - g_2 \frac{\partial B_1^{1,0}}{\partial \phi_2} \end{aligned}$$

where

$$(38) \quad D \equiv \omega_1 \frac{\partial}{\partial \phi_1} - \omega_2 \frac{\partial}{\partial \phi_2}$$

Similarly three components  $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3$  of  $\ddot{x}$  are

$$\ddot{x}_1 = D^2 B_1^{1,0}, \quad \ddot{x}_2 = D^2 B_2^{1,0}, \quad \ddot{x}_3 = D^2 B_3^{1,0} + 2\omega_1 f_2 \frac{\partial^2 B_1^{1,0}}{\partial \phi_1^2} - 2\omega_2 g_2 \frac{\partial^2 B_1^{1,0}}{\partial \phi_2^2}$$

In similar manner we can write the components of  $\dot{y}, \ddot{y}$ . Putting the values of  $x, y, \dot{x}, \dot{y}, \ddot{x}$  and  $\ddot{y}$  in terms of double d'Alembert series in equation ( 33) we get

$$(39) \quad \left( D^2 + 2E - 1 - \frac{3}{2}A_2 \right) B_2^{1,0} - \left\{ 2 \left( 1 + \frac{3}{4}A_2 \right) D - G \right\} B_2^{0,1} = X_2$$

$$(40) \quad \left\{ 2 \left( 1 + \frac{3}{4}A_2 \right) D + G \right\} B_2^{1,0} + \left( D^2 + 2F - 1 - \frac{3}{2}A_2 \right) B_2^{0,1} = Y_2$$

where

$$X_2 = \left[ \frac{\partial L_3}{\partial x} \right]_{x=B_1^{1,0}, y=B_1^{0,1}} \quad \text{and} \quad Y_2 = \left[ \frac{\partial L_3}{\partial y} \right]_{x=B_1^{1,0}, y=B_1^{0,1}}$$

These are two simultaneous partial differential equations in  $B_2^{1,0}$  and  $B_2^{0,1}$ . We solve these equations to find the values of  $B_2^{1,0}$  and  $B_2^{0,1}$ , from Eq. ( 39) and ( 40)

$$(41) \quad \Delta_1 \Delta_1 B_2^{1,0} = \Phi_2, \quad \Delta_1 \Delta_1 B_2^{0,1} = -\Psi_2 \quad \text{where} \quad \Delta_1 = D^2 + \omega_1^2, \Delta_2 = D^2 + \omega_2^2$$

$$(42) \quad \Phi_2 = (D^2 + 2F - n^2)X_2 + (2nD - G)Y_2$$

$$(43) \quad \Psi_2 = (2nD + G)X_2 - (D^2 + 2E - n^2)Y_2$$

The Eq.( 41) can be solved for  $B_2^{1,0}$  and  $B_2^{0,1}$  by putting the formula

$$\frac{1}{\Delta_1 \Delta_2} \begin{cases} \cos(p\phi_1 + q\phi_2) \\ \text{or} \\ \sin(p\phi_1 + q\phi_2) \end{cases} = \frac{1}{\Delta_1 \Delta_2} \begin{cases} \cos(p\phi_1 + q\phi_2) \\ \text{or} \\ \sin(p\phi_1 + q\phi_2) \end{cases}$$

where

$$\Delta_{p,q} = [\omega_1^2 - (\omega_1 p - \omega_2 q)^2] [\omega_2^2 - (\omega_1 p - \omega_2 q)^2]$$

provided  $\Delta_{p,q} \neq 0$ . Since  $\Delta_{1,0} = 0, \Delta_{0,1} = 0$  the terms  $\cos \phi_1, \sin \phi_1, \cos \phi_2, \sin \phi_2$  are the critical terms.  $\phi$  and  $\psi$  are free from such terms. By condition(1) of Moser's theorem  $k_1 \omega_1 + k_2 \omega_2 \neq 0$  for all pairs  $(k_1, k_2)$  of integers such that  $|k_1| + |k_2| \leq 4$ , therefore each of  $\omega_1, \omega_2, \omega_1 \pm 2\omega_2, \omega_2 \pm 2\omega_1$  is different from zero and consequently none of the divisors  $\Delta_{0,0}, \Delta_{0,2}, \Delta_{2,0}, \Delta_{1,1}, \Delta_{1,-1}$  is zero. The second order components  $B_2^{1,0}, B_2^{0,1}$  are as follows:

$$(44) \quad \begin{aligned} B_2^{1,0} = & r_1 I_1 + r_2 I_2 + r_3 I_1 \cos 2\phi_1 + r_4 I_2 \cos 2\phi_2 + r_5 I_1^{1/2} I_2^{1/2} \cos(\phi_1 - \phi_2) \\ & + r_6 I_1^{1/2} I_2^{1/2} \cos(\phi_1 + \phi_2) + r_7 I_1 \sin 2\phi_1 + r_8 I_2 \sin 2\phi_2 \\ & + r_9 I_1^{1/2} I_2^{1/2} \sin(\phi_1 - \phi_2) + r_{10} I_1^{1/2} I_2^{1/2} \sin(\phi_1 + \phi_2) \end{aligned}$$

and

$$(45) \quad \begin{aligned} B_2^{0,1} = & - \left\{ s_1 I_1 + s_2 I_2 + s_3 I_1 \cos 2\phi_1 + s_4 I_2 \cos 2\phi_2 + s_5 I_1^{1/2} I_2^{1/2} \cos(\phi_1 - \phi_2) \right. \\ & + s_6 I_1^{1/2} I_2^{1/2} \cos(\phi_1 + \phi_2) + s_7 I_1 \sin 2\phi_1 + s_8 I_2 \sin 2\phi_2 \\ & \left. + s_9 I_1^{1/2} I_2^{1/2} \sin(\phi_1 - \phi_2) + s_{10} I_1^{1/2} I_2^{1/2} \sin(\phi_1 + \phi_2) \right\} \end{aligned}$$

where

$$(46) \quad r_1 = \frac{1}{\omega_1^2 \omega_2^2} \left\{ J_{13}^2 \omega_1 F_4 + J_{13} J_{23} \omega_1 F_4' + \left( \frac{J_{21}^2}{\omega_1} + J_{23}^2 \omega_1 \right) F_4'' \right\}$$

$$(47) \quad r_2 = \frac{1}{\omega_1^2 \omega_2^2} \left\{ J_{14}^2 \omega_2 F_4 + J_{14} J_{24} \omega_2 F_4' + \left( \frac{J_{22}^2}{\omega_2} + J_{24}^2 \omega_2 \right) F_4'' \right\}$$

$$(48) \quad \begin{aligned} r_3 = & \frac{-1}{3\omega_1^2(4\omega_1^2 - \omega_2^2)} \left\{ 8\omega_1^3 J_{21}(J_{13}F_1' + 2J_{23}F_1'') + 4\omega_1^2 \left[ (J_{13}F_2 + J_{23}F_2'')J_{13}\omega_1 - \left( \frac{J_{21}^2}{\omega_1} - J_{23}^2\omega_1 \right) F_1'' \right] \right. \\ & \left. - 2\omega_1 J_{21}(J_{13}F_3' + 2J_{23}F_3'') - \omega_1 J_{13}(J_{13}F_4 + J_{23}F_4'')\omega_1 + \left( \frac{J_{21}^2}{\omega_1} - J_{23}^2\omega_1 \right) F_1'' \right\} \end{aligned}$$

$$(49) \quad \begin{aligned} r_4 = & \frac{1}{3\omega_2^2(4\omega_2^2 - \omega_1^2)} \left\{ 8\omega_2^3 J_{22}(J_{14}F_1' + 2J_{24}F_1'') - 4\omega_2^2 \left[ (J_{14}F_2 + J_{24}F_2'')J_{14}\omega_2 - \left( \frac{J_{22}^2}{\omega_2} - J_{24}^2\omega_2 \right) F_2'' \right] \right. \\ & \left. - 2\omega_2 J_{22}(J_{14}F_3' + 2J_{24}F_3'') - \omega_2 J_{14}(J_{14}F_4 + J_{24}F_4'')\omega_2 - \left( \frac{J_{22}^2}{\omega_2} - J_{24}^2\omega_2 \right) F_4'' \right\} \end{aligned}$$

$$\begin{aligned}
(50) \quad r_5 = & \frac{1}{\omega_1 \omega_2 (2\omega_1 + \omega_2)(4\omega_1 + 2\omega_2)} \left\{ (\omega_1 + \omega_2)^3 \left[ \left\{ J_{13} J_{22} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} - J_{14} J_{21} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} \right\} F'_1 \right. \right. \\
& - 2 \left\{ J_{21} J_{24} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} - J_{22} J_{23} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} \right\} F''_1 \Big] - (\omega_1 + \omega_2)^2 \left[ \left\{ 2 \left\{ J_{13} J_{14} F_2 \right. \right. \right. \\
& + (J_{13} J_{24} + J_{14} J_{23}) F'_2 \Big\} (\omega_1 \omega_2)^{1/2} + \left\{ \frac{J_{21} J_{22}}{(\omega_1 \omega_2)^{1/2}} + J_{23} J_{24} (\omega_1 \omega_2)^{1/2} \right\} F''_2 \Big] \\
& - (\omega_1 + \omega_2) \left[ \left\{ J_{13} J_{22} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} - J_{14} J_{21} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} \right\} F'_3 - 2 \left\{ J_{21} J_{24} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} - J_{22} J_{23} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} \right\} F''_3 \right] \\
& \left. + \left[ \left\{ 2 \left\{ J_{13} J_{14} F_4 + (J_{13} J_{24} + J_{14} J_{23}) F'_4 \right\} (\omega_1 \omega_2)^{1/2} + 2 \left\{ \frac{J_{21} J_{22}}{(\omega_1 \omega_2)^{1/2}} + J_{23} J_{24} (\omega_1 \omega_2)^{1/2} \right\} F''_4 \right\} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
(51) \quad r_6 = & \frac{-1}{\omega_1 \omega_2 (2\omega_1 - \omega_2)(4\omega_1 - 2\omega_2)} \left\{ (\omega_1 - \omega_2)^3 \left[ \left\{ J_{13} J_{22} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} - J_{14} J_{21} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} \right\} F'_1 \right. \right. \\
& + 2 \left\{ J_{21} J_{24} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} + J_{22} J_{23} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} \right\} F''_1 \Big] \\
& + (\omega_1 - \omega_2)^2 \left[ \left\{ 2 \left\{ J_{13} J_{14} F_2 + (J_{13} J_{24} + J_{14} J_{23}) F'_2 \right\} (\omega_1 \omega_2)^{1/2} \right. \right. \\
& - 2 \left\{ \frac{J_{21} J_{22}}{(\omega_1 \omega_2)^{1/2}} - J_{23} J_{24} (\omega_1 \omega_2)^{1/2} \right\} F''_2 \Big] - (\omega_1 - \omega_2) \left[ \left\{ J_{13} J_{22} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} \right. \right. \\
& - J_{14} J_{21} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} \Big\} F'_3 + 2 \left\{ J_{21} J_{22} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} + J_{22} J_{23} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} \right\} F''_3 \Big] \\
& \left. - \left[ \left\{ 2 \left\{ J_{13} J_{14} F_4 + (J_{13} J_{24} + J_{14} J_{23}) F'_4 \right\} (\omega_1 \omega_2)^{1/2} - 2 \left\{ \frac{J_{21} J_{22}}{(\omega_1 \omega_2)^{1/2}} - J_{23} J_{24} (\omega_1 \omega_2)^{1/2} \right\} F''_4 \right\} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
(52) \quad r_7 = & \frac{1}{3\omega_1^2(4\omega_1^2 - \omega_2^2)} \left\{ 8\omega_1^3 \left[ J_{13}(J_{13}F_1 + J_{23}F'_1)\omega_1 - \left( \frac{J_{21}^2}{\omega_1} - J_{23}^2\omega_1 \right) F''_1 \right] \right. \\
& - 2\omega_1 \left[ \omega_1 J_{13}(J_{13}F_3 + J_{23}F'_3) - \left( \frac{J_{21}^2}{\omega_1} - J_{23}^2\omega_1 \right) F''_3 \right] \\
& \left. - 4\omega_1^2 J_{21}(J_{13}F_2 + J_{23}F''_2)\omega_1 + J_{21}(J_{13}F'_4 + 2J_{23}F''_4) \right\}
\end{aligned}$$

$$\begin{aligned}
(53) \quad r_8 = & \frac{-1}{3\omega_2^2(4\omega_2^2 - \omega_1^2)} \left\{ 8\omega_2^3 \left[ J_{14}(J_{14}F_1 + J_{24}F'_1)\omega_2 - \left( \frac{J_{22}^2}{\omega_2} - J_{24}^2\omega_2 \right) F''_1 \right] \right. \\
& + 4\omega_2^2 J_{22}(J_{14}F_2 + 2J_{24}F''_2)\omega_2 - 2\omega_2 \left[ \omega_2 J_{14}(J_{14}F_3 + J_{24}F'_3) \right. \\
& \left. - \left( \frac{J_{22}^2}{\omega_2} - J_{24}^2\omega_2 \right) F''_3 \right] - J_{22}(J_{14}F'_4 + 2J_{24}F''_4) \Big\}
\end{aligned}$$

$$\begin{aligned}
(54) \quad r_9 = & \frac{1}{\omega_1 \omega_2 (2\omega_1 + \omega_2)(\omega_1 + 2\omega_2)} \left\{ (\omega_1 + \omega_2)^3 \left[ \{ 2J_{13}J_{14}F_1 \right. \right. \\
& + (J_{13}J_{24} + J_{14}J_{23})F'_1 \} (\omega_1 \omega_2)^{1/2} + 2 \left\{ \frac{J_{21}J_{22}}{(\omega_1 \omega_2)^{1/2}} + J_{23}J_{24}(\omega_1 \omega_2)^{1/2} \right\} F''_1 \Big] \\
& - (\omega_1 + \omega_2)^2 \left[ \left\{ J_{13}J_{22} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} - J_{14}J_{21} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} \right\} F'_2 \right. \\
& \left. - 2 \left\{ J_{21}J_{24} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} - J_{22}J_{23} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} \right\} F''_2 \right] \\
& - (\omega_1 + \omega_2) \left[ \left\{ 2 \left\{ J_{13}J_{14}F_3 + (J_{13}J_{24} + J_{14}J_{23})F'_3 \right\} (\omega_1 \omega_2)^{1/2} \right. \right. \\
& + 2 \left\{ \frac{J_{21}J_{22}}{(\omega_1 \omega_2)^{1/2}} + J_{23}J_{24}(\omega_1 \omega_2)^{1/2} \right\} F''_3 \Big] - \left[ \left\{ J_{13}J_{22} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} \right. \right. \\
& \left. \left. - J_{14}J_{21} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} \right\} F'_4 - 2 \left\{ J_{21}J_{24} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} - J_{22}J_{23} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} \right\} F''_4 \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
(55) \quad r_{10} = & \frac{1}{\omega_1 \omega_2 (2\omega_1 - \omega_2)(2\omega_2 - \omega_1)} \left\{ (\omega_1 - \omega_2)^3 \left[ \{ 2J_{13}J_{14}F_1 \right. \right. \\
& + (J_{13}J_{24} + J_{14}J_{23})F'_1 \} (\omega_1 \omega_2)^{1/2} - 2 \left\{ \frac{J_{21}J_{22}}{(\omega_1 \omega_2)^{1/2}} - J_{23}J_{24}(\omega_1 \omega_2)^{1/2} \right\} F''_1 \Big] \\
& - (\omega_1 - \omega_2)^2 \left[ \left\{ J_{13}J_{22} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} - J_{14}J_{21} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} \right\} F'_2 + 2 \left\{ J_{21}J_{24} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} + J_{22}J_{23} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} \right\} F''_2 \right] \\
& - (\omega_1 - \omega_2) \left[ \left\{ 2 \left\{ J_{13}J_{14}F_3 + (J_{13}J_{24} + J_{14}J_{23})F'_3 \right\} (\omega_1 \omega_2)^{1/2} - 2 \left\{ \frac{J_{21}J_{22}}{(\omega_1 \omega_2)^{1/2}} - J_{23}J_{24}(\omega_1 \omega_2)^{1/2} \right\} F''_3 \right. \right. \\
& \left. \left. + \left[ \left\{ J_{13}J_{22} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} - J_{14}J_{21} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} \right\} F'_4 + 2 \left\{ J_{21}J_{24} \left( \frac{\omega_2}{\omega_1} \right)^{1/2} - J_{22}J_{23} \left( \frac{\omega_1}{\omega_2} \right)^{1/2} \right\} F''_4 \right] \right\} \right]
\end{aligned}$$

We can write expressions of  $s_i$  with the help of  $r_i$  replacing  $F_i$  by  $G_i$ ,  $F'_i$  by  $G'_i$  and  $F''_i$  by  $G''_i$ , ( $i = 1, 2, 3, 4$ ) where

$$(56) \quad F_1 = \frac{-nW_1\epsilon}{6}$$

$$\begin{aligned}
(57) \quad F_2 = & \frac{3}{32} \left[ \frac{16}{3}\epsilon + 6A_2 - \frac{979}{18}A_2\epsilon + \frac{(143+9\gamma)}{6\sqrt{3}}nW_1 + \frac{(555+376\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\
& \left. + \gamma \left\{ 14 + \frac{4\epsilon}{3} + 25A_2 - \frac{1507}{18}A_2\epsilon - \frac{(215+29\gamma)}{6\sqrt{3}}nW_1 - \frac{2(1174+169\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right]
\end{aligned}$$

$$\begin{aligned}
(58) \quad F_3 = & \frac{3\sqrt{3}}{16} \left[ 14 - \frac{16}{3}\epsilon + \frac{23A_2}{2} - \frac{104}{9}A_2\epsilon + \frac{115(1+\gamma)}{18\sqrt{3}}nW_1 - \frac{2(439-68\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\
& \left. + \gamma \left\{ \frac{32\epsilon}{3} + 40A_2 - \frac{310}{9}A_2\epsilon + \frac{(511+53\gamma)}{6\sqrt{3}}nW_1 - \frac{(2519-249\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right]
\end{aligned}$$

$$\begin{aligned}
(59) \quad F_4 = & \frac{-3}{256} \left[ 364 + 420A_2 - \frac{17801A_2}{9}A_2\epsilon + \frac{(2821+189\gamma)}{3\sqrt{3}}nW_1 - \frac{(23077+9592\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\
& \left. + 28\gamma \left\{ 23 + \frac{100\epsilon}{21} + \frac{849A_2}{14} + \frac{59}{7}A_2\epsilon - \frac{(125+38\gamma)}{6\sqrt{3}}nW_1 - \frac{(87613-213\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right]
\end{aligned}$$

$$\begin{aligned}
(60) \quad F'_1 &= \frac{nW_1\epsilon}{3\sqrt{3}} \\
(61) \quad F'_2 &= \frac{3\sqrt{3}}{16} \left[ 14 - \frac{16}{3}\epsilon + A_2 - \frac{1367}{18}A_2\epsilon + \frac{115(1+\gamma)}{18\sqrt{3}}nW_1 - \frac{(863-136\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\
&\quad \left. + \gamma \left\{ \frac{32\epsilon}{3} + 40A_2 - \frac{382}{9}A_2\epsilon + \frac{(511+53\gamma)}{6\sqrt{3}}nW_1 - \frac{(2519-24\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right] \\
(62) \quad F'_3 &= \frac{-9}{8} \left[ \frac{8}{3}\epsilon + \frac{203A_2}{6} - \frac{721}{54}A_2\epsilon - \frac{(105+15\gamma)}{18\sqrt{3}}nW_1 - \frac{(319-114\gamma)}{81\sqrt{3}}nW_1\epsilon \right. \\
&\quad \left. + \gamma \left\{ 2 - \frac{4\epsilon}{9} - \frac{173A_2}{6} - \frac{781}{9}A_2\epsilon + \frac{(197+23\gamma)}{18\sqrt{3}}nW_1 - \frac{(265-32\gamma)}{81\sqrt{3}}nW_1\epsilon \right\} \right] \\
(63) \quad F'_4 &= \frac{-3\sqrt{3}}{16} \left[ 392 - \frac{532\epsilon}{3} + \frac{1918A_2}{3} - \frac{28582A_2}{9}A_2\epsilon + \frac{(203+1211\gamma)}{9\sqrt{3}}nW_1 + \frac{(949+4378\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\
&\quad \left. + 28\gamma \left\{ \frac{108\epsilon}{7} + \frac{4037A_2}{84} - \frac{611}{21}A_2\epsilon + \frac{(8397+919\gamma)}{84\sqrt{3}}nW_1 - \frac{(92266-1869\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right]
\end{aligned}$$

$$\begin{aligned}
(64) \quad F''_1 &= \frac{nW_1\epsilon}{6} \\
(65) \quad F''_2 &= \frac{-9}{32} \left[ \frac{8}{3}\epsilon + \frac{203A_2}{6} - \frac{625}{54}A_2\epsilon - \frac{(105+15\gamma)}{18\sqrt{3}}nW_1 - \frac{(307-114\gamma)}{81\sqrt{3}}nW_1\epsilon \right. \\
&\quad \left. + \gamma \left\{ 2 - \frac{4\epsilon}{9} + \frac{55A_2}{2} - \frac{797}{54}A_2\epsilon + \frac{(197+23\gamma)}{18\sqrt{3}}nW_1 - \frac{(211-32\gamma)}{81\sqrt{3}}nW_1\epsilon \right\} \right] \\
(66) \quad F''_3 &= \frac{-9\sqrt{3}}{16} \left[ 2 - \frac{8}{3}\epsilon + \frac{55A_2}{6} - \frac{134}{3}A_2\epsilon - \frac{(37+\gamma)}{18\sqrt{3}}nW_1 - \frac{(93+226\gamma)}{81\sqrt{3}}nW_1\epsilon \right. \\
&\quad \left. + \gamma \left\{ 4\epsilon + \frac{169}{27}A_2\epsilon + \frac{(241+45\gamma)}{18\sqrt{3}}nW_1 - \frac{(1558-126\gamma)}{81\sqrt{3}}nW_1\epsilon \right\} \right] \\
(67) \quad F''_4 &= \frac{9}{256} \left[ \frac{212}{3}\epsilon + \frac{2950A_2}{3} - \frac{1370A_2}{27}A_2\epsilon - \frac{(771+237\gamma)}{9\sqrt{3}}nW_1 - \frac{2(1907-984\gamma)}{81\sqrt{3}}nW_1\epsilon \right. \\
&\quad \left. + 28\gamma \left\{ \frac{11}{7} + \frac{4\epsilon}{9} - \frac{152A_2}{7} - \frac{36965}{504}A_2\epsilon + \frac{(2569+277\gamma)}{252\sqrt{3}}nW_1 + \frac{(22603+4396\gamma)}{1134\sqrt{3}}nW_1\epsilon \right\} \right]
\end{aligned}$$

$$\begin{aligned}
(68) \quad G_1 &= \frac{-nW_1\epsilon}{6} \\
(69) \quad G_2 &= \frac{3}{32} \left[ 14 - \frac{16}{3}\epsilon + A_2 - \frac{1367}{18}A_2\epsilon + \frac{115(1+\gamma)}{18\sqrt{3}}nW_1 - \frac{(863-136\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\
&\quad \left. + \gamma \left\{ \frac{32\epsilon}{3} + 40A_2 - \frac{382}{9}A_2\epsilon + \frac{(511+53\gamma)}{6\sqrt{3}}nW_1 - \frac{(2519-24\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right] \\
(70) \quad G_3 &= \frac{3\sqrt{3}}{16} \left[ \frac{16}{3}\epsilon + 6A_2 - \frac{907A_2}{18}A_2\epsilon + \frac{(143+9\gamma)}{6\sqrt{3}}nW_1 + \frac{(477+403\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\
&\quad \left. + \gamma \left\{ 14 + \frac{4\epsilon}{3} + \frac{71A_2}{2} - \frac{1489}{18}A_2\epsilon - \frac{(215+29\gamma)}{6\sqrt{3}}nW_1 - \frac{2(1174+169\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right] \\
(71) \quad G_4 &= \frac{3\sqrt{3}}{256} \left[ 84 + 52\epsilon + 212A_2 - 267A_2\epsilon + \frac{2(299+61\gamma)}{3\sqrt{3}}nW_1 - \frac{(14854+225\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\
&\quad \left. + \gamma \left\{ 32\epsilon + 156A_2 + 649A_2\epsilon - \frac{(562+8\gamma)}{3\sqrt{3}}nW_1 + \frac{(13285+5169\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right]
\end{aligned}$$

$$(72) \quad G'_1 = \frac{-nW_1\epsilon}{\sqrt{3}}$$

$$(73) \quad G'_2 = \frac{9}{16} \left[ \frac{8}{3}\epsilon + \frac{203A_2}{6} - \frac{625}{54}A_2\epsilon - \frac{(105+15\gamma)}{18\sqrt{3}}nW_1 - \frac{(307-114\gamma)}{81\sqrt{3}}nW_1\epsilon \right. \\ \left. - \gamma \left\{ 2 - \frac{4\epsilon}{9} - \frac{55A_2}{2} - \frac{797}{54}A_2\epsilon + \frac{(197+23\gamma)}{18\sqrt{3}}nW_1 - \frac{(211-32\gamma)}{81\sqrt{3}}nW_1\epsilon \right\} \right]$$

$$(74) \quad G'_3 = \frac{3\sqrt{3}}{8} \left[ 14 - \frac{16}{3}\epsilon + \frac{65A_2}{6} - \frac{1439}{18}A_2\epsilon + \frac{115(1+\gamma)}{18\sqrt{3}}nW_1 - \frac{(941-118\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\ \left. + \gamma \left\{ \frac{32\epsilon}{3} - 40A_2 - \frac{310}{9}A_2\epsilon + \frac{(511+53\gamma)}{6\sqrt{3}}nW_1 - \frac{(251-24\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right]$$

$$(75) \quad G'_4 = \frac{-9}{128} \left[ 12\epsilon - 287A_2 + \frac{847A_2}{9}A_2\epsilon - \frac{2(28+\gamma)}{\sqrt{3}}nW_1 - \frac{4(2210-69\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\ \left. - \gamma \left\{ 96 + \frac{152\epsilon}{3} + 135A_2 - \frac{2320}{9}A_2\epsilon + \frac{(497-123\gamma)}{3\sqrt{3}}nW_1 - \frac{4(17697+32\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right]$$

$$(76) \quad G''_1 = \frac{-nW_1\epsilon}{6}$$

$$(77) \quad G''_2 = \frac{9\sqrt{3}}{32} \left[ 2 - \frac{8}{3}\epsilon + \frac{23A_2}{3} - 44A_2\epsilon - \frac{(37+\gamma)}{18\sqrt{3}}nW_1 - \frac{(123+349\gamma)}{3\sqrt{3}}nW_1\epsilon \right. \\ \left. + \gamma \left\{ 4\epsilon + \frac{88A_2}{27} + \frac{(421+45\gamma)}{18\sqrt{3}}nW_1 - \frac{(1558-126\gamma)}{81\sqrt{3}}nW_1\epsilon \right\} \right]$$

$$(78) \quad G''_3 = \frac{-9}{16} \left[ \frac{8}{9}\epsilon + \frac{203A_2}{6} - \frac{589}{54}A_2\epsilon - \frac{5(51+2\gamma)}{18\sqrt{3}}nW_1 - \frac{(349-282\gamma)}{81\sqrt{3}}nW_1\epsilon \right. \\ \left. + \gamma \left\{ 2 - \frac{4\epsilon}{9} - 26A_2 - \frac{412}{27}A_2\epsilon + \frac{(197+23\gamma)}{18\sqrt{3}}nW_1 - \frac{(211-32\gamma)}{81\sqrt{3}}nW_1\epsilon \right\} \right]$$

$$(79) \quad G''_4 = \frac{-9\sqrt{3}}{256} \left[ 12 + \frac{20}{3}\epsilon + 76A_2 - \frac{350A_2}{3}A_2\epsilon + \frac{(32\gamma)}{3\sqrt{3}}nW_1 - \frac{2(1529+450\gamma)}{27\sqrt{3}}nW_1\epsilon \right. \\ \left. + \gamma \left\{ 8\epsilon - \frac{749A_2}{3} + \frac{808}{9}A_2\epsilon - \frac{(109-40\gamma)}{3\sqrt{3}}nW_1 + \frac{(35-1269\gamma)}{27\sqrt{3}}nW_1\epsilon \right\} \right]$$

#### 4. CONCLUSION

Using transformation  $x = B_1^{1,0} + B_2^{1,0}$  and  $y = B_1^{0,1} + B_2^{0,1}$  in Eq.( 13) the third order part  $H_3$  of the Hamiltonian in  $I_1^{1/2}I_2^{1/2}$  is of the form

$$(80) \quad H_3 = A_{3,0}I_1^{3/2} + A_{2,1}I_1I_2^{1/2} + A_{1,2}I_1^{1/2}I_2 + A_{0,3}I_2^{3/2}$$

We can verify that in Eq.( 80)  $A_{3,0}$  vanishes independently as in Deprit and Deprit Barhtolomé(1967). Similarly the other coefficients  $A_{2,1}, A_{1,2}, A_{0,3}$  are also found to be zero independently. Hence the third order part  $H_3$  of the Hamiltonian in  $I_1^{1/2}I_2^{1/2}$  is zero.

**Acknowledgment:** We are thankful to D.S.T. Government of India, New Delhi for sanctioning a project DST/MS/140/2K dated 02/01/2004 on this topic. We are also thankful to IUCAA Pune for providing financial assistance for visiting library and computer facility.

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